DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A2 Mathematical Methods II

TRINITY TERM 2019

THURSDAY, 25 APRIL 2019, 9.30am to 12.00pm

This exam paper contains three sections.
You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best answer(s), making a total of four answers.

Please start the answer to each question in a new answer booklet.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Nonlinear Systems

1. (a) [5 marks] Give the definition of a hyperbolic fixed point of system $\dot{x} = f(x), \ x \in \mathbb{R}^n$. Show that if $V(x)$ is a strict Lyapunov function for a fixed point $x = x_0$ (i.e. $\dot{V}(x) < 0$ for $x \in U(x_0)$-neighborhood of x_0 such that $x \neq x_0$) then there exists no closed orbit curve in this neighbourhood.

   (b) [4 marks] Consider the Lotka-Volterra predator-prey model

$$\begin{aligned} \dot{x} &= x(1-y), \qquad (1) \\ \dot{y} &= y(-\gamma + x), \qquad (2) \end{aligned}$$

   with $\gamma > 0$ and $x, y \geqslant 0$. By calculating the derivative dy/dx along the solutions calculate a general formula for trajectories in the form:

$$V(x,y) = \text{const.} \qquad (3)$$

   Show additionally that the axes $y = 0$ and $x = 0$ are invariant.

   (c) [6 marks] Find all the fixed points of (1)-(2). Identify their local stability and type (i.e. node, center, focus or saddle).

   [*Hint: To analyse stability of the non-hyperbolic point* $(\gamma,\, 1)$ *expand the conserved function (3) in the vicinity of this point.*]

   Sketch a phase plane portrait for (1)-(2) in the first quadrant $Q = \{(x,\, y)|x \geqslant 0,\, y \geqslant 0\}$.

   (d) [5 marks] Show that the function

$$\tilde{V}(x,y) = V(\gamma, 1) - V(x,y)$$

   with V found in (3) is a Lyapunov function for the fixed point $(\gamma,\, 1)$.

   (e) [5 marks] Consider now the extended Lotka-Volterra predator-prey model:

$$\begin{aligned} \dot{x} &= x(1 - \alpha x - y), \qquad (4) \\ \dot{y} &= y(-\gamma + x - \beta y), \qquad (5) \end{aligned}$$

   with $\alpha, \beta > 0$ and $\alpha\gamma < 1$. Identify the unique fixed point $\bar{x}$, $\bar{y}$ of (4)-(5) with $\bar{y} > 0$. Find a Lyapunov function for it and show that this point is asymptotically stable.

2. (a) [5 marks] Give the definition of a fixed point and a periodic orbit for the one-dimensional map $x_{n+1} = f(x_n)$. State a criterion for determining stability of a p-periodic map $\{x_1, x_2, ..., x_p\}$ in terms of the values of the derivative $f'(x_n)$, $n = 1, ..., p$.

(b) [7 marks] Find all the fixed points of the map

$$x_{n+1} = \frac{\lambda x_n}{1 + x_n^2}, \quad \lambda \in \mathbb{R} \setminus \{0\} \tag{6}$$

and determine their local stability.

(c) [8 marks] Find all the period-2 solutions of the map (6) and determine their local stability.

(d) [5 marks] Sketch the fixed points and period-2 solutions obtained in (b) and (c) in a bifurcation diagram versus the parameter $\lambda \in \mathbb{R} \setminus \{0\}$.

Section B: Further Mathematical Methods

3. (a) [15 marks] Solve the equation

$$y(x) = 1 - 2x + \lambda \int_0^1 (2xt^2 - t)y(t)\,\mathrm{d}t.$$

Be careful to consider all values of λ and give all solutions when multiple solutions exist.

(b) [10 marks] Let $f(x)$ satisfy

$$f\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2 + 2f^2 = \epsilon\left(\frac{\mathrm{d}f}{\mathrm{d}x} + xf^3\right), \qquad f \to 0 \text{ as } |x| \to \infty,$$

where $0 < \epsilon \ll 1$. By expanding $f = f_0 + \epsilon f_1 + \cdots$ show by direct substitution that the leading-order problem is satisfied by

$$f_0(x) = A\mathrm{e}^{-(x-a)^2},$$

for any A and a. Show that the homogeneous version of the first-order problem is satisfied by both $\partial f_0/\partial a$ and $\partial f_0/\partial A$. Deduce that $a = 0$ and determine the possible values of A.

4. (a) [12 marks] Suppose the function $u(t)$ is the optimal control which minimises the cost functional

$$C[x,u] = \int_0^T h(t,x,u)\,\mathrm{d}t$$

over all controls $u(t) \in C^1[0,T]$ satisfying the control problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x,u), \qquad x(0) = a, \quad x(T) = b,$$

where $\partial f/\partial u \neq 0$. Show that u satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h}{\partial u}\Big/\frac{\partial f}{\partial u}\right) = \frac{\partial h}{\partial x} - \frac{\partial f}{\partial x}\left(\frac{\partial h}{\partial u}\Big/\frac{\partial f}{\partial u}\right).$$

(b) [7 marks] A process obeys the control problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + u, \qquad x(0) = 1, \quad x(1) = 0,$$

and it is desired to minimise the integral

$$C[x,u] = \int_0^1 \left(u(t)^2 + 2tx(t)\right)\mathrm{d}t.$$

Show that

$$u = t - 1 - \frac{2\mathrm{e}^{1-t}}{\mathrm{e}+1},$$

and find the corresponding behaviour of x.

(c) [6 marks] Now suppose that the requirement $x(1) = 0$ is removed, leaving only the condition $x(0) = 1$. Show that the natural boundary condition at $t = 1$ is

$$\dot{x}(1) = x(1).$$

Show that the optimal solution for x is now

$$x = -t + \mathrm{e}^t,$$

and find the corresponding control u.

Section C: Further PDEs

5. (a) (i) [5 marks] Define the Hankel transform $\mathcal{H}[f(r);k]$ of a function $f(r)$. Show that

$$\mathcal{H}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r};k\right]=-k^2\mathcal{H}[f(r);k].$$

(ii) [9 marks] Consider a point source of fluid a distance a from an impermeable wall. The velocity potential $\phi(x,y)$ satisfies the equation

$$\nabla^2\phi=4\pi\delta(x)\delta(y)\delta(z),\qquad -\infty<x,y<\infty,\quad -a<z<\infty,$$

with $\partial\phi/\partial z=0$ on $z=-a$ and $\phi\to 0$ as $z\to\infty$. Show that

$$\phi=-2\int_0^\infty \mathrm{e}^{-ka}\cosh(k(z+a))J_0(kr)\,\mathrm{d}k$$

if $-a<z<0$ and find the corresponding expression for $z>0$.

(b) The Mellin transform is given by

$$\mathcal{M}[f(x);s]=F(s)=\int_0^\infty x^{s-1}f(x)\,\mathrm{d}x,$$

which exists in some strip $c_1<\mathrm{Re}(s)<c_2$. The inversion is given by

$$f(x)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}x^{-s}F(s)\,\mathrm{d}s,$$

where $c_1<c<c_2$.

(i) [3 marks] Show that for $a>0$, $\mathcal{M}[f(ax);s]=a^{-s}\mathcal{M}[f(x);s]$.

(ii) [8 marks] For $x>0$, let

$$G(x)=\sum_{k=1}^{\infty}k^2\mathrm{e}^{-k^{3/2}x}.$$

Use the Mellin transform to show that

$$G(x)\sim\frac{\alpha}{x^2}\qquad\text{as } x\to 0+,$$

where you should determine the constant α.

[*You may use without proof the fact that the gamma function*

$$\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

has poles at $x=-m$, $m=0,1,2,\dots$ with residue $(-1)^m/m!$. The Riemann zeta function, defined by

$$\zeta(x)=\sum_{n=1}^{\infty}n^{-x}$$

for Re$(x)>1$, may be analytically continued to a meromorphic function which has a single pole at $x=1$ with residue 1.]

6. Let the operator L be given by

$$Lu = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$$

on the interval $0 \leqslant x < \infty$ with the conditions $u(0) = 0$ and $u \in L^2[0, \infty)$.

(a) [10 marks] Show that for μ not a positive real number the Green's function for $Lu - \mu u$ is

$$G(x, \xi; \mu) = \begin{cases} \frac{1}{\sqrt{\mu}} \sin\left(\sqrt{\mu}\, x\right) \mathrm{e}^{\mathrm{i}\sqrt{\mu}\,\xi} & 0 \leqslant x < \xi < \infty, \\ \frac{1}{\sqrt{\mu}} \sin\left(\sqrt{\mu}\, \xi\right) \mathrm{e}^{\mathrm{i}\sqrt{\mu}\,x} & 0 \leqslant \xi < x < \infty, \end{cases}$$

where you should define the branch of the square root.

(b) [10 marks] Use this to find the corresponding spectral representation of the delta function. Show that the cases $x > \xi$ and $x < \xi$ both give the same result.

(c) [5 marks] The Fourier sine integral transform is defined, for suitable functions $f(x)$, by

$$F(t) = \int_0^\infty \sin(t\xi)\, f(\xi)\, \mathrm{d}\xi.$$

Deduce from (b) the inversion formula.